Productly linearly independent sequences

Jaroslav Hančl[1], **Katarína Korčeková** and **Lukáš Novotný**[2]

Department of Mathematics, University of Ostrava, 30. dubna 22, 701 03 Ostrava 1

e-mail: hancl@osu.cz, lukas.novotny@osu.cz, katarina.korcekova@seznam.cz

Abstract

We introduce the two new concepts, productly linearly independent sequences and productly irrational sequences. Then we prove a criterion for which certain infinite sequences of rational numbers are productly linearly independent. As a consequence we obtain a criterion for the irrationality of infinite products and a criterion for a sequence to be productly irrational.

Key words: Linear independence, infinite product.

1 Introduction

Using some techniques from Erdős [2] we prove

Theorem 1 Let K be a non-negative integer and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{1}{2^n} a_n = \limsup_{n\to\infty} a_n^{\frac{1}{(2K+3)^n}} = \infty.$$

Then the numbers $1, \prod_{n=1}^{\infty}(1+\frac{a_n+1}{a_n^2+1}), \prod_{n=1}^{\infty}(1+\frac{a_n+1}{na_n^2+1}), \cdots$, and $\prod_{n=1}^{\infty}(1+\frac{a_n+1}{n^K a_n^2+1})$ are linearly independent over the rational numbers.

We also prove a criterion for infinite products to be irrational.

Theorem 2 Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that

$$\lim_{n\to\infty} \frac{\log a_n}{\log n} = \limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty.$$

Then the number $\prod_{n=1}^{\infty}(1+\frac{a_n+1}{a_n^2+1})$ is irrational.

We do not know if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers with $\liminf_{n\to\infty} a_n^{\frac{1}{2^n}} > 1$ and such that the numbers $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ and $\prod_{n=1}^{\infty}(1+\frac{1}{a_n+1})$ are $\mathbb{Q}$-linearly dependent. But we know that if $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ and $a_n \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$ then the numbers $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ and $\prod_{n=1}^{\infty}(1+\frac{1}{a_n+1})$ are $\mathbb{Q}$-linearly independent. This is a consequence of the fact that $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})(\prod_{n=1}^{\infty}(1+\frac{1}{a_n+1}))^{-1} = \prod_{n=1}^{\infty}(1+\frac{1}{a_n(a_n+2)})$ and the theorem of Hančl and Kolouch [6] which shows that if $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ and $a_n \in \mathbb{Z}^+$ for all $n \in \mathbb{Z}^+$ then the number $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ is irrational.

A simple calculation shows that $\prod_{n=1}^{\infty}(1+\frac{1}{n(n+2)}) = 2$ and $\sum_{n=1}^{\infty}\frac{1}{n(n+2)} = \frac{3}{4}$. So the numbers $\prod_{n=1}^{\infty}(1+\frac{1}{n(n+2)})$ and $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ are rational, hence $\mathbb{Q}$-linearly dependent. On

[1]Supported by the grants no. ME09017 and MSM 6198898701

[2]Supported by the grant 01798/2011/RRC of the Moravian-Silesian region.

the other side we do not know if the numbers $\prod_{n=1}^{\infty}(1+\frac{1}{n^3})$ and $\sum_{n=1}^{\infty}\frac{1}{n^3}$ are $\mathbb{Q}$-linearly independent.

Let K be a positive integer greater then 1. It is not hard to prove that $\prod_{n=1}^{\infty}(1+\frac{1}{K^{2^n}})=\frac{K^2}{K^2-1}$, but we do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{(K^{2^n}+1)a_n})$ is irrational for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers. So it is natural to ask if the number $\sum_{n=1}^{\infty}\frac{1}{(K^{2^n}+1)a_n}$ is irrational for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers. The authors do not know the answer to this, and the case $K=2$ is an open problem of Erdős [3].

Hančl and Tijdeman [8] proved that the numbers 1, e, and $\sum_{n=1}^{\infty}\frac{[n^\alpha]}{n!}$ are $\mathbb{Q}$-linearly independent for all $\alpha\in\mathbb{R}^+$, $\alpha\notin\mathbb{Z}$. On the other side we do not know if the numbers $\prod_{n=1}^{\infty}(1+\frac{[n^\alpha]}{n!})$ are $\mathbb{Q}$-linearly independent for all $\alpha\in\mathbb{R}^+$. Moreover, we do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{n!})$ is irrational.

In 1975 Erdős [2] defined a so-called irrational sequence in the following way. We say that the sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers is irrational if for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the series $\sum_{n=1}^{\infty}\frac{1}{a_nc_n}$ is an irrational number. If $\{a_n\}_{n=1}^{\infty}$ is not an irrational sequence, then we say it is a rational sequence. Erdős also proved a theorem which gives a criterion for a sequence to be irrational in the same paper. Hančl [5] extended Erdős' definition to linear independence. For more information see [5] and [7]. Other criteria for linear independence can be found in [1] and [4].

Our main theorem is Theorem 3 which deals with $\mathbb{Q}$-linear independence. As a consequence of Theorem 3 we obtain a criterion for irrationality in Theorem 4.

2 Main results

In the spirit of Erdős we define the so-called productly linearly independent sequences in the following way.

Definition 1 Let $\{a_{i,n}\}_{n=1}^{\infty}$ for $i=1,\dots,K$ be sequences of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the numbers $\prod_{n=1}^{\infty}\big(1+\frac{1}{a_{1,n}c_n}\big)$, $\prod_{n=1}^{\infty}\big(1+\frac{1}{a_{2,n}c_n}\big)$, $\dots$, $\prod_{n=1}^{\infty}\big(1+\frac{1}{a_{K,n}c_n}\big)$, and 1 are $\mathbb{Q}$-linearly independent, then the sequences $\{a_{i,n}\}_{n=1}^{\infty}$ $i=1,\dots,K$ are said to be productly linearly independent.

We have a criterion for productly linearly independent sequences.

Theorem 3 Let K be a positive integer and ε, a, and b be real numbers such that $0<\varepsilon$, $0\le a$, $0\le b$, and $1-a-b>\frac{1}{1+\varepsilon}$. Suppose that $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$, $i=1,\dots,K$, are sequences of positive integers with $\{a_{1,n}\}_{n=1}^{\infty}$ non-decreasing and such that

$$\limsup_{n\to\infty} a_{1,n}^{\frac{1}{V^n}}=\infty\,,$$

where $V=\big(\frac{K+(K-1)a}{1-b-a}+1\big)$, and

$$\lim_{n\to\infty}\frac{a_{i,n}b_{j,n}}{b_{i,n}a_{j,n}}=0,\quad i,j=1,\dots,K,\quad i>j.$$

Assume that for every sufficiently large n

$$a_{1,n}\ge n^{1+\varepsilon},$$

$$b_{i,n}\le a_{1,n}^{\,b+\frac{1}{\log^{1+\varepsilon}\log a_{1,n}}},\quad i=1,\dots,K,$$

and

$$a_{1,n}^{1-(a+\frac{1}{\log^{1+\varepsilon}\log a_{1,n}})} \le a_{i,n} \le a_{1,n}^{1+a+\frac{1}{\log^{1+\varepsilon}\log a_{1,n}}}, \quad i = 1, \ldots, K.$$

Then the sequences $\{\frac{a_{i,n}}{b_{i,n}}\}_{n=1}^{\infty}$ $i = 1, \ldots, K$, are productly linearly independent.

Example 1 The sequences

$$\left\{\frac{n^{6\cdot 9^n}+5}{n^{9^n}+3}\right\}_{n=1}^{\infty} \qquad \text{and} \qquad \left\{\frac{n^{3\cdot 9^n}+7}{n^{8^n}+5}\right\}_{n=1}^{\infty}$$

are productly linearly independent.

In a similar way we can define the so-called productly irrational sequences.

Definition 2 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the number $\sum_{n=1}^{\infty}(1+\frac{1}{a_n c_n})$ is irrational, then the sequence $\{a_n\}_{n=1}^{\infty}$ is said to be productly irrational.

For productly irrational sequences we have the following criterion.

Theorem 4 Let ε and b be real numbers such that $0 \le b < 1 - \frac{1}{1+\varepsilon}$ and $0 < \varepsilon$. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are sequences of positive integers with $\{a_n\}_{n=1}^{\infty}$ non-decreasing and such that $\limsup_{n\to\infty} a_n^{\left(\frac{1-b}{2-b}\right)^n} = \infty$. Assume that $a_n \ge n^{1+\varepsilon}$ and $b_n \le a_n^{b+\frac{1}{\log^{1+\varepsilon}\log a_n}}$ hold for every sufficiently large n. Then the sequence $\{\frac{a_n}{b_n}\}_{n=1}^{\infty}$ is productly irrational and the number $\prod_{n=1}^{\infty}(1+\frac{b_n}{a_n})$ is irrational.

Example 2 The sequences

$$\left\{\frac{n^{2\cdot 3^n}+n^2}{n^{3^n}+n}\right\}_{n=1}^{\infty} \qquad \text{and} \qquad \left\{\frac{3^{3\cdot 4^n}+n!}{3^{2\cdot 4^n}+n^3}\right\}_{n=1}^{\infty}$$

are productly irrational.

3 Proofs

Proof: (of Theorem 1) Theorem 1 is an immediate consequence of Theorem 3 when we set $K := K+1$, $a_{j,n} := n^{j-1}a_n^2+1$ and $b_{j,n} := n^{j-1}a_n^2+1$ for all $n \in \mathbb{Z}^+$ and $j \in \{1, \cdots, K+1\}$, $a := 0$, and $b := \frac{1}{2}$.

□

Proof: (of Theorem 2) Theorem 2 is an immediate consequence of Theorem 4 when we set $a_n := a_n^2 + 1$ and $b_n := a_n + 1$ for all $n \in \mathbb{Z}^+$, and $b := \frac{1}{2}$.

□

Proof: (of Theorem 4) Theorem 4 is an immediate consequence of Theorem 3 when we set $K := 1$, $a_{1,n} := a_n$, and $b_{1,n} := b_n$ for all $n \in \mathbb{Z}^+$, $a := 0$, and $b := b$.

□

Proof: (of Example 1) Example 1 is an immediate consequence of Theorem 3 when we set $K := 2$, $a_{1,n} := n^{6\cdot 9^n}+5$, $b_{1,n} := n^{9^n}+3$, $a_{2,n} := n^{3\cdot 9^n}+7$, and $b_{2,n} := n^{8^n}+5$ for all $n \in \mathbb{Z}^+$, $a := \frac{1}{2}$, and $b := \frac{1}{6}$.

□

Proof: (of Example 2) Example 2 is an immediate consequence of Theorem 4 when we set $a_n := n^{2\cdot 3^n} + n^2$ and $b_n := n^{3^n} + n$ for all $n \in \mathbb{Z}^+$, and $b := \frac{1}{2}$ for the first sequence, and $a_n := 3^{3\cdot 4^n} + n!$ and $b_n := 3^{2\cdot 4^n} + n^3$ for all $n \in \mathbb{Z}^+$, and $b := \frac{2}{3}$ for the second sequence.

□

Proof: (of Theorem 3) We can see that other Theorems are consequences of Theorem 3, but the proof is difficult and long. So we introduced here only idea of this proof. To prove that the numbers $\prod_{n=1}^{\infty}\left(1+\frac{b_{1,n}}{a_{1,n}}\right)$, $\dots$, $\prod_{n=1}^{\infty}\left(1+\frac{b_{K,n}}{a_{K,n}}\right)$, and the number 1 are linearly independent over the rational numbers we will prove that for every K-tuple of integers $\beta_1, \beta_2, \dots, \beta_K$ (not all equal to zero) the sum

$$\beta = \sum_{i=1}^{K} \beta_i \prod_{n=1}^{\infty}\left(1+\frac{b_{i,n}}{a_{i,n}}\right) \tag{1}$$

is an irrational number. We will suppose that $\beta = \frac{p}{q}$ is a rational number. From this and (1) we obtain that for every $N \in \mathbb{Z}^+$ the number

$$\beta(N) = \Big|\prod_{n=1}^{N-1}\prod_{i=1}^{K} a_{i,n}\Big(p - q\sum_{i=1}^{K}\beta_i \prod_{n=1}^{N-1}\big(1+\frac{b_{i,n}}{a_{i,n}}\big)\Big)\Big|$$

is an integer. We will show that $0 < \beta(N) < 1$ for all sufficiently large N and it will be a contradiction with rationality of β.

□

References

[1] Deajim A., Siksek S.: On the $\mathbb{Q}$-linear independence of the sums $\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n!}$, J. Number Theory 131, no. 4, (2011), 745–749.

[2] Erdős P.: Some Problems and Results on the Irrationality of the Sum of Infinite Series, J. Math. Sci. 10 (1975), 1-7.

[3] Erdős P.: Erdős problem no. 6, 1995 Prague Midsummer Combinatorial Workshop, KAM Series (95-309), M. Klazar (ed.), (1995), page 5.

[4] Erdős P. and Straus E. G. : On the irrationality of certain series, Pacific journal of mathematics, Vol.55, No.1, 1974, 85-92.

[5] J. Hančl: Linearly unrelated sequences, Pacific Journal of Mathematics, Vol.190, No.2, (1999), 299-310.

[6] Hančl J., Kolouch O.: Erdős' method for determining the irrationality of products. (to appear).

[7] J. Hančl, S. Sobková: A general criterion for linearly unrelated sequences, Tsukuba Journal of Mathematics, Vol.27, No.2, (2003), 341-357.

[8] Hančl J., Tijdeman R.: On the irrationality of factorial series. Acta Arith. 118, no. 4, (2005), 383–401.