A criterion for linear independence of infinite products

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Abstract:

Using an idea of Erdős the paper establishes a criterion for the linear independence of infinite products which consist of rational numbers. A criterion for irrationality is obtained as a consequence.

Keywords: *Linear independence; infinite products.*

Introduction

Following Erdős [3] we prove

Theorem 1 Let K be a non-negative integer and let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that

$$1 < \liminf_{n\to\infty} a_n^{\frac{1}{(K+2)^n}} < \limsup_{n\to\infty} a_n^{\frac{1}{(K+2)^n}} < \infty.$$

Then the numbers $1, \prod_{n=1}^{\infty}(1+\frac{1}{a_n+1}), \prod_{n=1}^{\infty}(1+\frac{1}{na_n+1}), \cdots, and \prod_{n=1}^{\infty}(1+\frac{1}{n^K a_n+1})$ are linearly independent over the rational numbers.

As a consequence of this theorem we obtain a criterion for infinite products to be irrational.

Theorem 2 Let $\{a_n\}_{n=1}^{\infty}$ be a non-decreasing sequence of positive integers such that

$$1 < \liminf_{n\to\infty} a_n^{\frac{1}{2^n}} < \limsup_{n\to\infty} a_n^{\frac{1}{2^n}} < \infty.$$

Then the number $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ is irrational.

The authors do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{2^{2^n}a_n+1})$ is irrational for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers although we know from another theorem of Erdős [3] that the number $\sum_{n=1}^{\infty}\frac{1}{2^{2^n}a_n}$ is irrational for every sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers. Hančl and Kolouch [8] proved that if $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ and $a_n \in \mathbb{Z}^+$ then the number $\prod_{n=1}^{\infty}(1+\frac{1}{a_n})$ is irrational, but we do not know if it is transcendental.

It is not difficult to prove that $\prod_{n=1}^{\infty}(1+\frac{1}{2^{2^n}}) = \frac{4}{3}$, but we do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{(2^{2^n}+1)a_n})$ is irrational for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers. Erdős [4] asked if the number $\sum_{n=1}^{\infty}\frac{1}{(2^{2^n}+1)a_n}$ is irrational for all sequences $\{a_n\}_{n=1}^{\infty}$ of positive integers.

A simple calculation shows that $\prod_{n=2}^{\infty}(1-\frac{1}{n^2}) = \frac{1}{2}$. On the other side the authors are not able to decide if the number $\prod_{n=1}^{\infty}(1+\frac{1}{n^2})$ is irrational. In fact we are not able to prove that the number $\prod_{n=1}^{\infty}(1+\frac{1}{n^k})$ is irrational for any $k \in \mathbb{Z}^+$, $k \neq 1$. This is analogous to the problem of the irrationality of the function $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} = \prod_{n=1}^{\infty}(1+\frac{1}{p_n^k-1})$ for

[1]Supported by the grants no. ME09017 and MSM 6198898701

[2]Supported by the grant 01798/2011/RRC of the Moravian-Silesian region.

$k \in \mathbb{Z}^+$, $k \neq 1$ where $\{p_n\}_{n=1}^{\infty}$ is the increasing sequence of all primes. We know that for even k the number $\zeta(k)$ is transcendental and that $\zeta(3)$ is an irrational number. But we do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{p_n^k})$ is irrational for any $k \in \mathbb{Z}^+$, $k \neq 1$.

Hančl and Tijdeman [10] proved that if $\{v_n\}_{n=1}^{\infty}$ is an unbounded non-decreasing sequence of positive integers such that $v_n = o(\sqrt{n})$, then the numbers 1 and $\sum_{n=1}^{\infty} \frac{n^{tv_n}}{n!}$, where $t \in \mathbb{N}$, are $\mathbb{Q}$-linearly independent. On the other side we do not know if the numbers $\prod_{n=1}^{\infty}(1+\frac{1}{n!})$ and $\prod_{n=1}^{\infty}(1+\frac{n^{\pi(n)}}{n!})$ are $\mathbb{Q}$-linearly independent where $\pi(n)$ denotes the number of primes less than or equal to n. Moreover, we do not know if the number $\prod_{n=1}^{\infty}(1+\frac{1}{n!})$ is irrational.

Erdős and Straus [5] proved the $\mathbb{Q}$-linear independence of the numbers 1, $\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$, $\sum_{n=1}^{\infty} \frac{\phi(n)}{n!}$, and $\sum_{n=1}^{\infty} \frac{b_n}{n!}$ where $b_n \in \mathbb{Z}$ and $b_n < n^{\frac{1}{2}-\varepsilon}$ for all large n and where $\sigma(n)$ and $\phi(n)$ denote the sum of divisors of n, and Euler's totient function of n respectively. A nice result proved recently is due to Deajim and Siksek [2]. They proved the $\mathbb{Q}$-linear independence of the numbers $\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$ where $\sigma_k(n)$ denotes the sum of the k-th powers of the positive divisors of n, assuming a standard conjecture of Schinzel on the prime values taken by a family of polynomials. Several general results concerning the linear independence of infinite series can be found in [7] and [9]. For other results in this theory see [6], [11], [12] or [14], for instance.

Choi and Zhou [1] obtained some results regarding the linear independence of infinite products of rational numbers. There exists a nice book by Nishioka [13] which contains a review of results concerning the linear and algebraic independence of infinite products and series which use the strong tools of Mahler's method.

Our main theorem is Theorem 3 concerning $\mathbb{Q}$-linear independence. As a consequence of Theorem 3 we obtain a criterion for irrationality in Theorem 4.

We denote by $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{N}$, and $\mathbb{Q}$ the set of all integers, positive integers, natural numbers including zero, and rational numbers, respectively. The functions $\pi(x)$, $[x]$, and $\log_2 x$ are the number of primes less than or equal to x, the greatest integer less than or equal to x, and the logarithm to the base 2 of the number x, respectively.

Main result

Our first theorem is a basic result which deals with the $\mathbb{Q}$-linear independence of infinite products of rational numbers.

Theorem 3 Let K be a positive integer and let ε be a positive real number. Assume that $\{a_{i,n}\}_{n=1}^{\infty}$ and $\{b_{i,n}\}_{n=1}^{\infty}$ $(i = 1, ..., K)$ are sequences of positive integers such that $\{a_{1,n}\}_{n=1}^{\infty}$ is non-decreasing,

$$\liminf_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} < \limsup_{n\to\infty} a_{1,n}^{\frac{1}{(K+1)^n}} < \infty \tag{1}$$

and

$$\lim_{n\to\infty} \frac{a_{i,n} b_{j,n}}{b_{i,n} a_{j,n}} = 0, \qquad \text{for all} \quad j, i \in \{1, ..., K\}, \quad i > j. \tag{2}$$

Suppose that for every sufficiently large number n

$$b_{i,n} < a_{1,n}^{\frac{1}{\log_2^{1+\varepsilon} \log_2 a_{1,n}}}, \qquad i = 1, ..., K, \tag{3}$$

$$a_{1,n} \geq n^{1+\varepsilon}, \tag{4}$$

and

$$a_{i,n}a_{1,n}^{-\frac{1}{\log_2^{1+\varepsilon}\log_2 a_{1,n}}} < a_{1,n} < a_{i,n}a_{1,n}^{\frac{1}{\log_2^{1+\varepsilon}\log_2 a_{1,n}}}, \qquad i = 2, ..., K. \tag{5}$$

Then the products $\prod_{n=1}^{\infty}(1+\frac{b_{1,n}}{a_{1,n}})$,..., and $\prod_{n=1}^{\infty}(1+\frac{b_{K,n}}{a_{K,n}})$, and the number 1 are $\mathbb{Q}$-linearly independent.

Example 1 Let K be a positive integer greater than 1. As an immediate consequence of Theorem 3 we obtain that the products

$$\prod_{n=1}^{\infty}\left(1+\frac{7^{j2^{\pi([n/4])}}+n^n}{2^{K^{2^{[\log_2 n]}}}+5^n}\right),$$

where $j = 1, ..., K-1$, are $\mathbb{Q}$-linearly independent.

Our second theorem is a consequence of the previous theorem and deals with the irrationality of infinite products over the rational numbers

Theorem 4 Let ε be a positive real number. Suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers with $\{a_n\}_{n=1}^{\infty}$ non-decreasing and such that $\liminf_{n\to\infty} a_n^{\frac{1}{2^n}} < \limsup_{n\to\infty} a_n^{\frac{1}{2^n}} < \infty$. Assume that $a_n \geq n^{1+\varepsilon}$ and $b_n \leq a_n^{\log_2^{-(1+\varepsilon)}\log_2 a_n}$ hold for every large n. Then the product $\prod_{n=1}^{\infty}(1+\frac{b_n}{a_n})$ is an irrational number.

Example 2 From Theorem 4 we obtain that the numbers

$$\prod_{n=1}^{\infty}\left(1+\frac{5^{\pi(n)}+1}{2^{2^{2^{2^{[\log_2\log_2 n]}}}}+n}\right) \quad \text{and} \quad \prod_{n=1}^{\infty}\left(1+\frac{2^{\pi(n)}+5}{2^{2^{2^{2^{[\log_2\log_2 n]}}}}+2\pi(n)}\right)$$

are irrational.

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